XV. Fluid Physics

SPACE SCIENCES DIVISION

A. Viscous and Inviscid Amplification Rates of Two- and Three-Dimensional Linear Disturbances in the Laminar Compressible Boundary Layer, *L. M. Mack*

In SPS 37-36, Vol. IV, pp. 221–223, the inviscid stability of the laminar boundary layer was computed for three-dimensional disturbances for free-stream Mach numbers M_1 between 4.5 and 10.0. It was found that the most unstable first-mode disturbance is a three-dimensional wave with a wave angle σ (σ is the angle between the wave normal and the free-stream direction) between 50 and 60 deg. The maximum time rate of amplification of the most unstable three-dimensional disturbance in that Mach number range is roughly twice that of the most unstable two-dimensional disturbance. In contrast, the most unstable second and higher-mode disturbances are two dimensional. These predictions were subsequently confirmed in their essential points by the experiments of Kendall at $M_1 = 4.5$ (SPS 37-39, Vol. IV, pp. 147–148).

The experimental results of Laufer–Vrebalovich (Ref. 1) at $M_1 = 2.2$, which have been available for several years, do not agree with the two-dimensional stability theory in either the location of the upper branch of the neutral-stability curve or the maximum rate of amplification. Experimentally, the latter is about 10 times the theoretical value. The disagreement in the neutral-stability curves led Brown (Ref. 2) to abandon the parallel-flow theory and include in the basic equations all of the terms involving v, the mean vertical velocity in the boundary layer, but without any $\partial/\partial x$ terms. When these equations, in three-dimensional form, were solved for a 55-deg wave, the agreement between the theoretical and experimental neutral-stability curves of frequency was markedly improved.

In spite of this agreement, some fundamental questions still remain. First, it is not known if the eigenvalues, α (the wave number in the x direction) and c_r (the phase velocity), and the amplification rate αc_i of Brown's theory agree with the experimental results. Second, it is not known if the experimental disturbances were or were not three dimensional, or, if they were, whether the component with a 55-deg angle was dominant. The artificial disturbances were intended to be two dimensional, but this point was not checked experimentally. The agreement of the neutral-stability curves and amplification rates for natural and artificial disturbances was used by Laufer and Vrebalovich to conclude that two- and three-dimensional disturbances have the same stability characteristics. However, equally valid interpretations would be that both the natural and artificial disturbances were either two dimensional or three dimensional. The latter possibility must be regarded as the more probable. Dunn and Lin (Ref. 3) concluded on theoretical grounds that up to about $M_1 = 1.8$, three-dimensional disturbances are no more unstable than two-dimensional disturbances for an insulated-wall boundary layer. This conclusion was based solely on an examination of the critical Reynolds number, which can be a misleading criterion, particularly for disturbances at different wave angles.

In this paper, the parallel-flow theory is used to study the stability characteristics of three-dimensional disturbances at lower Mach numbers than in SPS 37-36, Vol. IV. Since this form of theory gives satisfactory results at $M_1 = 4.5$, it must at least be thoroughly tested at $M_1 = 2.2$ before being abandoned for something more complicated. It might be mentioned that Brown's computation with the mean v equations of a neutral-stability curve at $M_1 = 5.0$ for a 55-deg wave gives an instability region with the upper-branch point at a frequency less than half that measured by Kendall at $M_1 = 4.5$ for the same wave angle.

The inviscid-theory calculations (SPS 37-36, Vol. IV) were extended from $M_1 = 4.5$ down to 1.8, with the results shown in Fig. 1, where the maximum time rate of amplification (for any wave number or any angle) is given as a function of Mach number for both two- and three-dimensional first-mode disturbances. The wave angles of the most unstable disturbances, to the nearest 5 deg, are designated on the figure. The ranges of the experimental measurements at both $M_1 = 2.2$ (Laufer–Vrebalovich) and $M_1 = 4.5$ (Kendall, corrected for boundary-layer thickness) are also shown. R is the square root of the x Reynolds number. It is seen that the ratio of three-dimensional to two-dimensional amplification rates becomes quite large at low Mach numbers. At $M_1 = 1.8$, the maximum amplification rate of the most unstable three-dimensional wave is 130 times that of the two-dimensional wave; at $M_1 = 2.2$, it is 33 times larger than for the two-dimensional wave; and at $M_1 = 3.0$, the ratio of the maximum amplification rates is reduced to 5.8.

The reason for the enhanced instability of three-dimensional disturbances at low Mach numbers is not

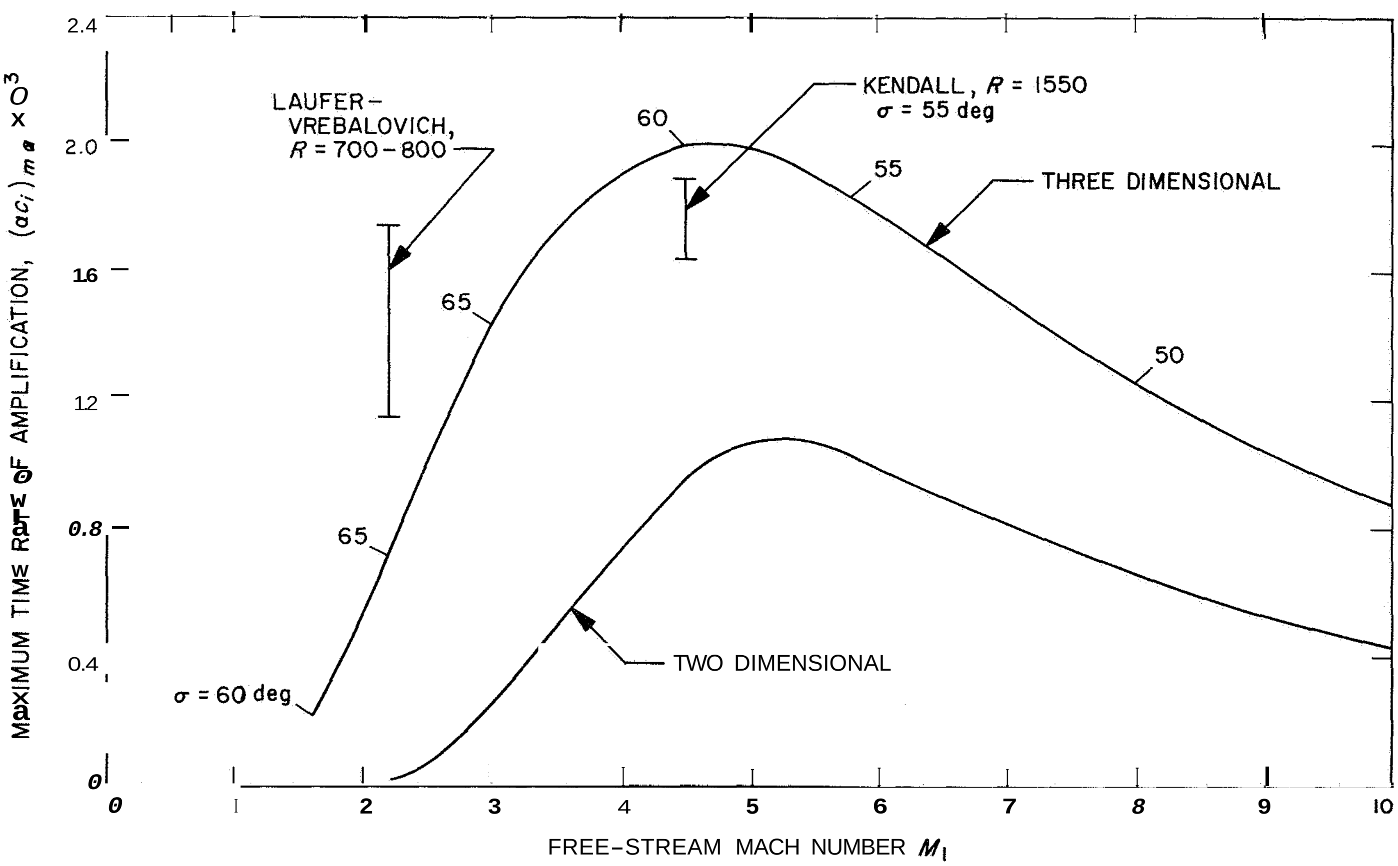

Fig. 1. Inviscid first-mode maximum amplification rates versus Mach number for two- and three-dimensional disturbances

hard to find. The phase velocity of an amplified disturbance is always larger than $c_0 = 1 - (1/M_1)$, the sonic limit, and less than c_s, which is equal to the mean velocity at η_s, the generalized inflection point, where $(u'/T)' = 0$. At the lowest Mach numbers ($M_1 < 1$), c_s is small and the boundary layer is almost stable to inviscid disturbances. Above $M_1 = 1$, c_s increases rapidly, but the instability of a two-dimensional disturbance remains small because c_0 is almost equal to c_s. It is only for $M_1 > 3$, that $c_s - c_0$ becomes sufficiently large for an appreciable two-dimensional instability to develop. For a three-dimensional disturbance, c_0 can always be reduced to zero for a sufficiently large wave angle (small Mach number normal to wave front), and c_s is independent of σ. Consequently, there is no lower bound to c_r other than zero, and the boundary layer can demonstrate what might be called its natural instability.

The inviscid maximum amplification rate at $M_1 = 2.2$ for a three-dimensional wave differs from the experimental values by only a factor of about 2. Whether this improved agreement is further improved or worsened at finite Reynolds numbers depends on the effect of viscosity at this Mach number. For a two-dimensional disturbance, the destabilizing action of viscosity, which is solely responsible for boundary-layer instability at $M_1 = 0$, decreases 'sharply with increasing Mach number, as shown in Fig. 2, until at $M_1 = 2.6$ the maximum amplification at low Reynolds number is only a local maximum and is equal to the inviscid amplification rate. At Mach numbers above 3, even the local maximum disappears and the action of viscosity is stabilizing over the Reynolds number range of interest; i.e., a decrease of Reynolds number leads to a decrease in the maximum amplification rate. If the action of viscosity is similar for three-dimensional

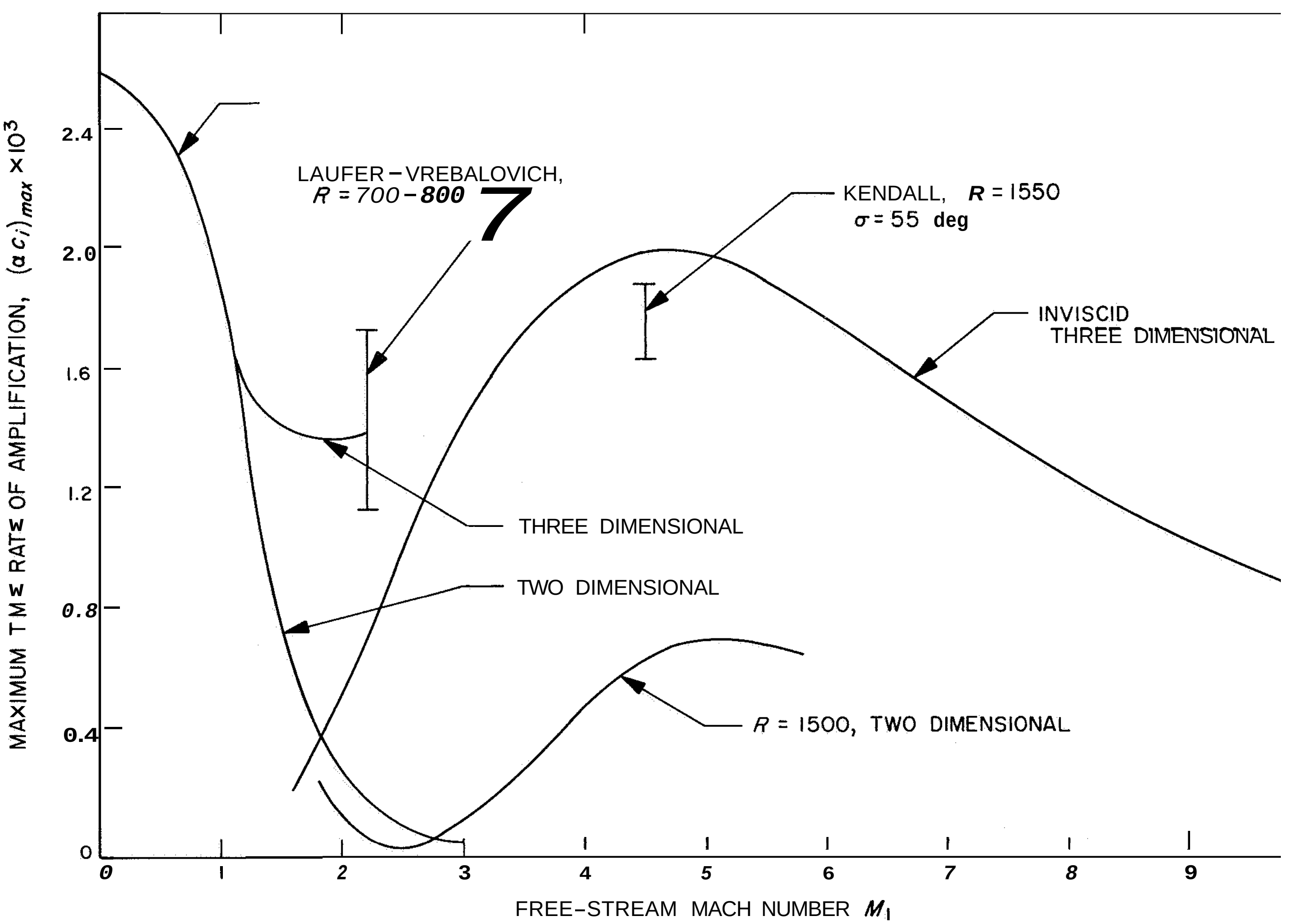

Fig. 2. Viscous first-mode maximum amplification rates versus Mach number for two- and three-dimensional disturbances

disturbances, the amplification rates at finite Reynolds numbers for Mach numbers around 2 will be larger than the inviscid rate, and for $M_1 > 3$ they will be smaller than the inviscid rate. Consequently, the possibility exists that a three-dimensional disturbance has amplification rates in agreement with experiment at both $M_1 = 2.2$ and 4.5.

It is no simple matter to compute the amplification rates of three-dimensional disturbances at finite Reynolds numbers. At $M_1 = 0$, the three-dimensional stability equations are transformed by the Squire transformation into the two-dimensional Orr–Sommerfield equation for the same boundary-layer profile. The three-dimensional amplification rates can be obtained from the two-dimensional rates already obtained. Further, the two-dimensional disturbance has the largest amplification rate. The generalization of the Squire transformation used by Dunn and Lin, which transforms the three-dimensional inviscid equations (but not the solutions) into two-dimensional equations, does not similarly transform the complete parallel-flow viscous equations. The Dunn–Lin equations, which include only what are supposed to be the leading viscous terms, do transform, as would any set of equations that do not include the dissipation terms in the energy equation. But if all the parallel-flow viscous terms are to be kept, then it is necessary to include an additional momentum equation, as Brown did for the equations with the v terms. The system of equations is increased to eighth order from the present sixth order, and the computer program must be rewritten.

Within the framework of the sixth-order system, at least three courses of action are possible: First, use the Dunn–Lin equations; second, use a set of equations which are as close as possible to the correct equations; third, use the present complete equations as if all the terms transformed. The second option is perhaps logically superior to the others, but has not yet been done. Both the first and third options have been carried out, with most of the computations performed with the complete equations. Brown has computed a neutral-stability curve at $M_1 = 2.2$ for a wave angle of 55 deg from both the two-dimensional complete equations (including mean v terms) plus the Squire transformation, and from the eighth-order system. With the v terms included, there is a considerable increase in the number of terms that do not transform. Even so, the largest error in the Reynolds number of a neutral-stability point at a fixed frequency is about 25%.

Fig. 2 gives the results obtained at finite Reynolds numbers together with the first-mode three-dimensional inviscid curve and experimental results repeated from Fig. 1. Both the two- and three-dimensional results obtained with the complete equations are shown. The two-dimensional disturbance is the most unstable up to about $M_1 = 1$. For higher Mach numbers, the three-dimensional disturbances ($\sigma = 45$ deg at $M_1 = 1.3$, $\sigma = 60$ deg at $M_1 = 2.2$) are the most unstable, with the action of viscosity up to at least $M_1 = 2.2$ plainly destabilizing as for two-dimensional disturbances. Further, the amplification rate at $M_1 = 2.2$ of the most unstable three-dimensional disturbance is within the range of the experimental results.

A few comparisons have been made between amplification rates computed from the Dunn–Lin equations and the complete equations. At $M_1 = 1.3$ the two-dimensional neutral-stability curves computed from the two sets of equations are close together. Table 1 shows that the amplification rates for a 50-deg wave are also in good agreement at a Reynolds number of 1600. At $M_1 = 2.2$, where there is an important difference between the two-dimensional neutral-stability curves, a large percentage difference is evident in the two-dimensional amplification rates. For the 60-deg wave, the arithmetic difference of the two amplification rates is the same as for the two-dimensional wave, but the percentage difference is small. The most important comparison, with the amplification rate computed from the eighth-order system, is not available. However, the good agreement between the results obtained from the Dunn–Lin and complete equations indicates that where the inviscid instability is important, as it is for three-dimensional waves at $M_1 = 2.2$, the leading viscous terms of Dunn's analysis are truly dominant, and the additional viscous terms, some of which do not transform by the Squire transformation, play only a secondary role in establishing the amplification rate.

Table 1. Comparison of amplification rates computed from the two-dimensional Dunn–Lin and complete viscous equations

M_1	σ, deg	α	R	Dunn–Lin Eqs. $\alpha c_i \times 10^3$	Complete Eqs. $\alpha c_i \times 10^3$
1.3	50	0.060	1600	1.50	1.46
2.2	0	0.045	600	0.30	0.18
	60	0.040	1200	1.49	1.39

References

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